Second Midterm Exam–Version 1

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- **There are four problems in this exam and some have multiple parts. The points add up to 100.**
- Your solutions must be fully uploaded within 2 hr 15 minutes from the start, or before 5:45PM, whichever comes first. Be mindful of the passage of time so that you do not miss the opportunity to scan and upload your work.
- You are allowed to use the course materials: lecture notes, materials on Canvas, Durrett's textbook and your own notes to solve the problems. No other materials or outside help is allowed. In particular, you cannot communicate with anyone else or look up solutions on the internet.
- Simplify your answer if it is possible with reasonable effort.
- Solutions without justification cannot get credit. *Cross out the writing that you do not wish to be graded on.*

Problem	Points
1	/30
2	/20
3	/20
4	/30
Total	/100

1. Nick eats lunch at the campus food court every day. Each day he eats one of **Burger**, **Salad**, and **Pizza**. Let X_n denote his meal on the nth day for $n \geq 0$. Assume that his daily choices obey a Markov chain with transition matrix

$$\begin{array}{c} \\ \mathbf{B} \\ \mathbf{S} \\ \mathbf{P} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{B} & \mathbf{S} & \mathbf{P} \end{array} \\ \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/2 & 1/6 & 1/3 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}.$$

Note that ther are self-arcs so the MC is aperiodic and the MC convergence theorem holds.

(a) (10 pts) What is the long term fraction of days on which Nick eats a Burger?

(b) (10 pts) Find the limits of $P_B(X_n = S)$ and $P_B(X_n = B, X_{n+2} = S)$ as $n \to \infty$.

(c) (10 pts) Assume the prices of B, S, and P are \$5, \$3, and \$6, respectively. Find the long term average cost per day of Nick's lunch.

a) First find invariant distribution

$$\pi = \pi \mathbb{P} = (\pi(B)\ \pi(S)\ \pi(P)) \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/2 & 1/6 & 1/3 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \left(\tfrac{2}{3}\pi(B) + \tfrac{1}{2}\pi(S) + \tfrac{1}{2}\pi(P),\ \tfrac{1}{3}\pi(B) + \tfrac{1}{6}\pi(S) + \tfrac{1}{2}\pi(P),\ \tfrac{1}{3}\pi(S) \right)$$

$$\begin{cases} \frac{1}{3}\pi(B) = \frac{1}{2}\pi(S) + \frac{1}{2}\pi(P) \\ \frac{5}{6}\pi(S) = \frac{1}{3}\pi(B) + \frac{1}{2}\pi(P) \\ \pi(P) = \frac{1}{3}\pi(S) \end{cases} \longrightarrow \pi(B) = \frac{5}{6}\pi(S)$$

also $\pi(B) + \pi(S) + \pi(P) = 1$ so $\left(\frac{5}{6} + 1 + \frac{1}{3}\right)\pi(S) = 1 \to \pi(S) = \frac{6}{13}$

$$\pi = \left[\frac{5}{13}, \frac{6}{13}, \frac{2}{13} \right]$$

By the MC convergence theorem, the fraction of days that Nick eats a Burger $P_\mu(X_n = B) \to \pi(B)$ as $n \to \infty$. So $\boxed{P_\mu(X_n = B) \xrightarrow[n\to\infty]{} \frac{5}{13}}$

also could use SLLN for MCs with $f(x) = \begin{cases} 1, & x = B \\ 0, & x \neq B \end{cases}$ then $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f(X_k) = \pi(B)$

b) $P_B(X_n = S) \to \pi(S)$ by MC convergence thm

$$\boxed{P_B(X_n = S) \to \frac{6}{13}}$$

$$P_B(X_n = S, X_{n+2} = S) = P_B(X_n = S)\, p^2(S,S) \xrightarrow[n\to\infty]{} \pi(S)\, p^{(2)}(S,S) = \pi(S)\big(p(S,S)p(S,S) + p(S,B)\,p(B,S)\, p(S,P)\,p(P,S)\big)$$

$$= \frac{6}{13}\left(\frac{1}{6}\frac{1}{6} + \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{2} \right)$$

$$\boxed{P_B(X_n = S, X_{n+2} = S) \to \frac{4}{13}}$$

c)
$$f(X_n) = \begin{cases} 5, & X_n = B \\ 3, & X_n = S \\ 6, & X_n = P \end{cases}$$

By SLLN for finite MCs,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = \sum_{x \in S} f(x)\pi(x)$$

$$= f(B)\pi(B) + f(S)\pi(S) + f(P)\pi(P)$$

$$= 5\cdot\frac{5}{13} + 3\cdot\frac{6}{13} + 6\cdot\frac{2}{13}$$

$$\boxed{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = \frac{55}{13}}$$

2. Roll a fair six-sided die **five** times. Let X denote the number of threes among the outcomes, and let Y denote the number of twos among the outcomes.

(a) (15 pts) Find the conditional expectation $E[Y|X]$.

(b) (5 pts) Find the conditional expectation $E[X^2Y|X]$.

a) $$E[Y|X=x] = \sum_K K\, P(Y=K|X=x)$$

$$= \sum_K K \cdot \frac{P(Y=k, X=x)}{P(X=x)}$$

Let $Z = X+Y$ $\quad P(Y=k, X=x) = P(Z=k+x)$

$$E[Y|X=x] = \sum_K K \frac{P(Z=K+x)}{P(X=x)}$$

$X \sim Binom(5, \frac{1}{6})$

$Z \sim Binom(5, \frac{1}{3})$

$$= \sum_K K \frac{\binom{5}{k+x}\left(\frac{1}{3}\right)^{K+x}\left(\frac{2}{3}\right)^{5-K-x}}{\binom{5}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{5-x}}$$

$$= \sum_K K \frac{x!(5-x)!}{(K+x)!(5-K-x)!} \frac{\left(\frac{1}{2}\right)^{k+x}\left(\frac{1}{6}\right)^{K+x}\left(\frac{4}{5}\right)^{5-K-x}\left(\frac{5}{6}\right)^{5-K-x}}{\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{5-x}}$$

$$= \sum_K K \frac{(5-x)(5-x-1)\cdots(5-k-x+1)}{(k+x)(K+x-1)\cdots(x+1)} \left(\frac{1}{2}\right)^{K+x}\left(\frac{1}{6}\right)^K\left(\frac{4}{5}\right)^{5-K-x}\left(\frac{5}{6}\right)^{-k}$$

⋮

Easier Way: $E[Y|X]$ tells us we know there are X threes. There fore they can not be twos. $E[Y|X]$ will be the same as $E[\tilde{Y}]$ if we only roll $N-X = 5-X$ times with a success probabity of $\frac{1}{5}$ (since we know the # of 3's the remaining set is $\{1,2,4,5,6\}$)

$E[Y|X] = E[\tilde{Y}]$ where $\tilde{Y} \sim Binom(5-X, \frac{1}{5})$

$= np$ (binomial expectation)

$$\boxed{E[Y|X] = \frac{1}{5}(5-X)}$$

Check: if $X=5$, $E[Y|X] = 0$ ✓

if $X=0$, $E[Y|X] = \frac{1}{5}$ ✓

b) $E[X^2Y|X] = X^2 E[Y|X]$ since X^2 is a function of the conditioning variable

Plug in $E[Y|X]$ from part a.

$$\boxed{E[X^2Y|X] = \frac{1}{5}X^2(5-X)}$$

3. Let $X_1, X_2, X_3, \ldots$ be i.i.d. random variables with possible values $\{0, 1, 2\}$ and uniform distribution

$$P(X_1 = 0) = P(X_1 = 1) = P(X_1 = 2) = \frac{1}{3}.$$

$E[X_1] = \frac{1}{3} + \frac{2}{3} = 1$

(a) (10 pts) Define the process V_n by

$$V_1 = X_1 \quad \text{and} \quad V_n = X_n X_{n-1} \quad \text{for } n \geq 2.$$

$E[2^{X_1}] = \frac{1}{3}2^0 + \frac{1}{3}2^1 + \frac{1}{3}2^2 = \frac{7}{3}$

Decide whether V_n is or is not a martingale with respect to $\{X_k\}_{k\geq 1}$.

(b) (10 pts) Find a constant c such that the process

$$M_0 = 1, \qquad M_n = c^n\, 2^{X_1+\cdots+X_n} \quad \text{for } n \geq 1,$$

is a martingale with respect to $\{X_k\}_{k\geq 1}$.

a) 1) V_n is a function of $(X_0 \cdots X_n)$ since it depends only on X_n and X_{n-1}. ✓

2) $E[V_n]$ is finite since X_n is finite ✓

3) $E[(V_{n+1} - V_n) | X_0 \cdots X_n] = E[(X_{n+1}X_n - X_n X_{n-1}) | X_0 \cdots X_n]$

$= X_n(E[X_{n+1} | X_0 \cdots X_n] - X_{n-1})$ using linearity and properties of cond. expectation.

$= X_n(E[X_{n+1}] - X_{n-1})$ by independence

$= X_n(1 - X_{n-1})$

$\neq 0$ so $E[V_{n+1} | X_0 \cdots X_n] \neq V_n$ and $\boxed{V_n \text{ is not a Martingale w.r.t. } \{X_k\}}$

b) 1) M_n depends only on $X_0 \cdots X_n$ ✓

2) $E[M_n]$ is finite if $|c|<1$ ✓ (and possibly other values)

3) $E[(M_{n+1} \quad) | X_0 \cdots X_n] = E[(c^{n+1} 2^{X_1 + \cdots + X_{n+1}} | X_0 \cdots X_n]$

$= c^{n+1} 2^{X_1 + \cdots + X_n} E[2^{X_{n+1}}]$ by linearity and independence

$= \frac{7}{3}c(c^n 2^{X_1 + \cdots + X_n})$

For the third property to hold, set $\frac{7}{3}c(c^n 2^{X_1 + \cdots + X_n}) = c^n 2^{X_1 + \cdots + X_n}$

so $\frac{7}{3}c = 1 \rightarrow \boxed{c = \frac{3}{7}}$

$|c|<1$ so the second property holds. Thus $\boxed{M_n \text{ is a martingale w.r.t. } \{X_k\}}$

4. For $p \in (0,1)$, consider a branching process with offspring distribution $\beta = \{\beta_k\}_{0 \le k < \infty}$ given by

$$P(Z = k) = \beta_k = (k+1)(1-p)^2 p^k, \quad k = 0, 1, 2, \ldots$$

Let $\pi = \pi(p)$ denote the extinction probability of this branching process, as a function of the parameter p. The probability generating function of this distribution β is $g(s) = \left(\frac{1-p}{1-ps}\right)^2$ for $|s| < \frac{1}{p}$.

(a) (10 pts) Derive the expected value of Z from the given $g(s)$. (Other calculations of $E[Z]$ get half the credit.)

(b) (10 pts) For what values of p does the branching process go extinct with probability 1?

(c) (10 pts) Compute the extinction probability π for the case $p = \frac{1}{2}$.

a) $g'(s) = 2\left(\frac{1-p}{1-ps}\right)\frac{(-1)(-p)(1-p)}{(1-ps)^2} = \frac{2p(1-p)^2}{(1-ps)^3}$

$E[Z] = g'(1^-) = \frac{2p(1-p)^2}{(1-p)^3}$

$\boxed{E[Z] = \frac{2p}{1-p}}$

b) If $E[Z] \le 1$ then $\pi = 1$ (subcritical/critical case)

so $\frac{2p}{1-p} \le 1$

$2p \le 1-p$

$3p \le 1$

$\boxed{p \le \frac{1}{3}}$

c) $\pi = g(\pi) = \left(\frac{1-p}{1-p\pi}\right)^2$

$(1-p)^2 = \pi(1-p\pi)^2$

$= \pi(p^2\pi^2 - 2p\pi + 1)$

$0 = p^2\pi^3 - 2p\pi^2 + \pi - (1-p)^2$

When $p = \frac{1}{2} \rightarrow 0 = \frac{1}{4}\pi^3 - \pi^2 + \pi - \frac{1}{4}$

$0 = \frac{1}{4}(x-1)(x^2-3x+1)$

roots are $x = 1, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Take the smallest value so $\boxed{\pi = \frac{3}{2} - \frac{\sqrt{5}}{2}}$