Assigned: Wednesday, April 3, 2019
Due: Wednesday, April 10, 2019

MATH 322: HOMEWORK 10

NOTE:

- NO LATE homework will be accepted.
- Homework is due at the BEGINNING of class.
- Solutions to computer problems should be uploaded to Canvas before class.
- Solutions to pencil-and-paper problems should be handed in at the beginning of class.
- Your solutions must be presented in a way that is legible, space efficient, and easy to understand. You will lose points if you do not accomplish this.

1. Mixed boundary conditions.

   Consider the heat equation with mixed boundary conditions:

$$u_t = \kappa u_{xx}, \qquad \text{PDE} \tag{1}$$
$$u(x,0) = g(x), \qquad \text{IC} \tag{2}$$
$$u(0,t) = u_x(L,t) = 0, \qquad \text{BC} \tag{3}$$

   where the boundary condition is Dirichlet at $x = 0$ and Neumann at $x = L$.

   (a) Recall from class that, for the heat equation, separation of variables leads to $T(t) = e^{-\beta^2 t}$ and $X(x) = \cos \beta x$ or $\sin \beta x$.

   Find the values of β for which $\cos \beta x$ and/or $\sin \beta x$ satisfies the mixed boundary conditions.

   This should give you a collection of an infinite number of functions of the form $\cos \beta x$ and/or $\sin \beta x$ for different values of β.

   (b) Sketch a graph of the first 3 of your functions.

   (c) Show that your functions are orthogonal.

   (d) What is the solution $u(x,t)$ to the initial–boundary-value problem for the heat equation with initial condition $g(x) = \sin \frac{\pi x}{2L}$?

   What did we learn here?

   Separation of variables also applies to cases with mixed boundary conditions. The key is to determine which values of β are appropriate for $\sin \beta x$ and/or $\cos \beta x$.

2. Heat equation with forcing.

   Consider the heat equation with forcing:

$$u_t = \kappa u_{xx} + f(x,t), \qquad \text{PDE} \tag{4}$$
$$u(x,0) = 0, \qquad \text{IC} \tag{5}$$
$$u_x(0,t) = u_x(L,t) = 0, \qquad \text{BC} \tag{6}$$

   where Neumann boundary conditions are specified. For a forcing function of

$$f(x,t) = e^{-t}\sin^2\frac{\pi x}{L}, \tag{7}$$

   determine the solution $u(x,t)$.

   What did we learn here?

   Separation of variables can also be applied to problems with forcing. The forcing f and solution u can both be written as Fourier series in x to allow $u(x,t)$ to be determined.

3. Heat equation with non-homogeneous boundary conditions.

   Consider the heat equation with non-homogeneous boundary conditions:

$$u_t = \kappa u_{xx}, \qquad \text{PDE} \tag{8}$$
$$u(x,0) = 0, \qquad \text{IC} \tag{9}$$
$$u(0,t) = h(t), \qquad \text{BC} \tag{10}$$
$$u(L,t) = 0. \qquad \text{BC} \tag{11}$$

   Consider a homogenizing transformation of the form

$$u(x,t) = v(x,t) + w(x,t). \tag{12}$$

   (a) Determine a simple function $w(x,t)$ that satisfies the boundary conditions.

   (b) Find the PDE, IC, and BCs that $v(x,t)$ must satisfy.
   [But do not solve for $v(x,t)$.]

   What did we learn here?

   Separation of variables can also be applied to problems with non-homogeneous boundary conditions. A homogenizing transformation can be used to transform the problem into a different problem with homogeneous boundary conditions, and with forcing, which we know how to solve.

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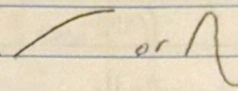
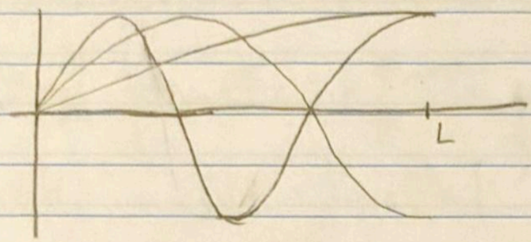
1) a) $u(0,t)=0$ requires sin solutions.

$u_x(L,t)=0$ requires $1/4$ or $3/4$ wavelengths i.e. 

$\cancel{\boxed{\beta = \frac{n\pi}{L} + \frac{\pi}{2L}}}$ $\quad n=0,1,2,3\cdots$ $\quad$ or $\quad \boxed{\beta = \frac{n\pi}{2L}}$ $\quad$ n is odd ✓

b)



✓

c) $\displaystyle\int_0^L \sin\left(\frac{n\pi x}{2L}\right)\sin\left(\frac{m\pi x}{2L}\right)dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \end{cases}$ work? -5

d) $u(x,t) = \sum_{n=0}^{\infty} \hat{u}_n e^{-\beta^2 kt} \sin(\beta x)$

$$= \sum_{n=0}^{\infty} \hat{u}_n e^{-\left(\frac{n\pi}{2L}\right)^2 kt} \sin\left(\frac{n\pi}{2L}x\right)$$

at $t=0$,

$$\sin\frac{\pi x}{2L} = \sum_{n=0}^{\infty} \hat{u}_n \sin\left(\frac{n\pi}{2L}x\right)$$

$\hat{u}_1 = 1$, others $= 0$

$$\boxed{u(x,t) = e^{-\left(\frac{\pi}{2L}\right)^2 kt} \sin\left(\frac{\pi}{2L}x\right)}$$ ✓

2) $u_t = k u_{xx} + f(x,t)$

$$f(x,t) = e^{-t}\sin^2\frac{\pi x}{L} = e^{-t}\left(1-\cos^2\left(\frac{\pi x}{L}\right)\right)$$
$$= e^{-t}\left(1-\frac{1}{2}-\frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)\right)$$
$$= e^{-t}\left(\frac{1}{2}-\frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)\right)$$

So $\hat{f}_0 = 1$, $\hat{f}_2 = \frac{-1}{2}$ all others $= 0$

$$f(x,t) = \frac{1}{2}\hat{f}_0 + \sum_{n=1}^{\infty}\hat{f}_n \cos\frac{n\pi}{L}x$$

$$u(x,t) = \frac{1}{2}\hat{u}_0 + \sum_{n=1}^{\infty}\hat{u}_n \cos\frac{n\pi}{L}x$$

$$\partial_t\left(\frac{1}{2}\hat{u}_0 + \sum_{n=1}^{\infty}\hat{u}_n\cos\frac{n\pi}{L}x\right) = k\,\partial_x^2\left(\frac{1}{2}\hat{u}_0 + \sum_{n=1}^{\infty}\hat{u}_n\cos\frac{n\pi}{L}x\right) + \frac{1}{2}\hat{f}_0 + \sum_{n=1}^{\infty}\hat{f}_n\cos\frac{n\pi}{L}x$$

$$\frac{1}{2}\frac{d\hat{u}_0}{dt} + \sum_{n=1}^{\infty}\frac{d\hat{u}_n}{dt}\cos\frac{n\pi}{L}x = \underbrace{k\sum_{n=1}^{\infty}-\left(\frac{n\pi}{L}\right)^2\hat{u}_n\cos\frac{n\pi}{L}x}_{} + \underbrace{\frac{1}{2}\hat{f}_0 + \sum_{n=1}^{\infty}\hat{f}_n\cos\frac{n\pi}{L}x}_{}$$

$$\int_0^L \cos\frac{m\pi}{L}x\left[\quad\right]dx = \int_0^L \cos\frac{m\pi}{L}x\left[\quad\right]dx + \int_0^L \cos\frac{m\pi}{L}x\left[\quad\right]dx$$

$$\frac{d\hat{u}_n}{dt} = -k\left(\frac{n\pi}{L}\right)^2\hat{u}_n + \hat{f}_n$$

Solved in class: $\hat{u}_n(t) = \int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 k(t-t')}\hat{f}_n(t')\,dt'$ since $\hat{g}_n = 0$

Since only $\hat{f}_0$ and $\hat{f}_2 \neq 0$, only $\hat{u}_0$ and $\hat{u}_2 \neq 0$ so

$$u(x,t) = \frac{1}{2}\hat{u}_0 + \hat{u}_2\cos\left(\frac{2\pi}{L}x\right)$$

$$\hat{u}_0 = \int_0^t e^0\,dt' = t$$

$$\hat{u}_2 = \int_0^t -e^{\left(\frac{2\pi}{L}\right)^2 k(t-t')}\cdot\frac{1}{2}\,dt' = \frac{1}{2}\left[\frac{-1}{-\left(\frac{2\pi}{L}\right)^2 k}e^{\left(\frac{2\pi}{L}\right)^2 k(t-t')}\right]_0^t$$

$$= \frac{1}{2\left(\frac{2\pi}{L}\right)^2 k}\left[1-e^{\left(\frac{2\pi}{L}\right)^2 kt}\right]$$

$$\boxed{u(x,t) = \frac{1}{2}\left[t + \frac{L^2}{4\pi^2 k}\left(1-e^{\left(\frac{2\pi}{L}\right)^2 kt}\right)\cos\left(\frac{2\pi}{L}x\right)\right]}$$

3) $u_t = k u_{xx}$ PDE
$u(x,0) = 0$ IC
$u(0,t) = h(t)$ BC
$u(L,t) = 0$ BC

$u = v + w$ $\quad w = \frac{-h(t)}{L}x + h(t)$ $\quad \boxed{w = h(t)\left(1 - \frac{x}{L}\right)}$

v satisfies $\quad v_t + w_t = k(v_{xx} + \cancel{w_{xx}}^{0})$

$$v_t + h'(t)\left(1 - \frac{x}{L}\right) = k v_{xx}$$

$$\boxed{v_t = k v_{xx} - h'(t)\left(1 - \frac{x}{L}\right)}$$ forced PDE with $f(x,t) = -h'(t)\left(1 - \frac{x}{L}\right)$

Write $f(x,t)$ as fourier series

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{L}\right) \qquad f_n(t) = \frac{2}{L}\int_0^L f(x,t)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$= \frac{2}{L}\int_0^L -h'(t)\left(1 - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$f_n(t) = \frac{2}{L}\left[-h'(t)\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx - n\pi x\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L - n\pi\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\right]$$

⋮ don't have to solve

$\boxed{\text{v satisfies homogeneous BCs with the same IC as u}}$