**Assigned:** Wednesday, April 10, 2019
**Due:** Wednesday, April 17, 2019

---

MATH 322: HOMEWORK 11

**NOTE:**

- NO LATE homework will be accepted.
- Homework is due at the BEGINNING of class.
- Solutions to computer problems should be uploaded to Canvas before class.
- Solutions to pencil-and-paper problems should be handed in at the beginning of class.
- Your solutions must be presented in a way that is legible, space efficient, and easy to understand. You will lose points if you do not accomplish this.

1. Wave equation.

   Consider the wave equation on the finite interval $(0, L)$:

$$u_{tt} - c^2 u_{xx} = 0, \qquad \text{PDE} \qquad (1)$$

$$u_x(0,t) = u_x(L,t) = 0, \qquad \text{BC} \qquad (2)$$

   where Neumann boundary conditions are specified.

   Physically, with Neumann boundary conditions, $u(x,t)$ could represent the height of a fluid that sloshes between two walls.

   (a) Find the general Fourier series solution by repeating the derivation from class, now considering Neumann instead of Dirichlet boundary conditions. Your final solution should be

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty} \left( A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L} \right) \cos\frac{n\pi x}{L}. \qquad (3)$$

   (b) Consider the following *general* initial conditions:

$$u(x,0) = g(x), \qquad \text{IC} \qquad (4)$$

$$u_t(x,0) = h(x). \qquad \text{IC} \qquad (5)$$

   Derive formulas that relate the Fourier coefficients $A_n, B_n, \hat{g}_n, \hat{h}_n$.

   (c) Consider the following *specific* initial conditions:

$$u(x,0) = 1 - \frac{1}{2}\cos\frac{2\pi x}{L}, \qquad \text{IC} \qquad (6)$$

$$u_t(x,0) = 0. \qquad \text{IC} \qquad (7)$$

   Find the solution $u(x,t)$.

(d) Make a surface plot to illustrate the solution $u(x, t)$. Use values of $L = 1$ and $c = 1$, and plot the solution for times $t = 0$ to $t = 1$.

Use Matlab or a similar computing language to generate the plot. Upload two types of files to Canvas: (i) a graphics file (e.g., Graphic1.pdf) of your plot, and (ii) a file (e.g., Code1.m) of the code you used to generate the plot.

What did we learn here?

Separation of variables provides a second option, in addition to Green's functions, for solving the wave equation on a finite interval.

For the wave equation, the amplitude of each Fourier mode oscillates in time, in contrast to the case of the heat equation, which has amplitudes that decay in time.

2. Laplace's equation.

Consider Laplace's equation on the rectangle with $0 < x < L$ and $0 < y < H$:

$$u_{xx} + u_{yy} = 0, \qquad \text{PDE} \qquad (8)$$

$$u(x, 0) = 0, \qquad \text{BC} \qquad (9)$$

$$u(x, H) = g(x), \qquad \text{BC} \qquad (10)$$

$$u_x(0, y) = u_x(L, y) = 0, \qquad \text{BC} \qquad (11)$$

where a mixture of Dirichlet and Neumann boundary conditions is specified, and only one of the sides has a boundary condition that is nonhomogeneous.

(a) Find the general solution $u(x, y)$, written as an infinite series, by repeating the derivation from class, but with modifications to account for the different boundary conditions used here.

[In particular, notice two differences from the derivation in class: (i) the nonhomogeneous boundary here is on the $y = H$ side, and (ii) it is the "$x$ direction" here that has homogeneous boundary conditions, at $x = 0$ and $x = L$.]

(b) Find the solution $u(x, y)$ that arises in the case when $g(x) = \cos(\pi x/L)$.

(c) Make a surface plot to illustrate the solution $u(x, t)$. Use values of $L = H = 1$.

Use Matlab or a similar computing language to generate the plot. Upload two types of files to Canvas: (i) a graphics file (e.g., Graphic2.pdf) of your plot, and (ii) a file (e.g., Code2.m) of the code you used to generate the plot.

What did we learn here?

Separation of variables provides a second option, in addition to Green's functions, for solving Laplace's equation on a finite interval.

For Laplace's equation in 2D, the basis functions oscillate in one direction (either $x$ or $y$) and grow/decay in the other direction.

3. Multiple choice.

   Consider the following Fourier series on $(0, L)$:

   $$g(x) = x(1 - x) = \sum_{n=1}^{\infty} \hat{g}_n \sin \frac{n\pi x}{L}. \tag{12}$$

   The Fourier series will converge in which sense?

   (a) Uniform.
   (b) Pointwise.
   (c) Mean-square ($L^2$).

   The answer could be zero, one, two, or all three. Select as many as are appropriate.

4. Multiple choice.

   Consider the following Fourier series on $(0, L)$:

   $$g(x) = \frac{1}{2} - \left| x - \frac{1}{2} \right| = \sum_{n=1}^{\infty} \hat{g}_n \sin \frac{n\pi x}{L}. \tag{13}$$

   The Fourier series will converge in which sense?

   (a) Uniform.
   (b) Pointwise.
   (c) Mean-square ($L^2$).

   The answer could be zero, one, two, or all three. Select as many as are appropriate.

5. Short-answer problem.

   Consider a Fourier series on $(0, L)$:

   $$g(x) = \sum_{n=1}^{\infty} \hat{g}_n \sin \frac{n\pi x}{L}. \tag{14}$$

   Give an example of a function $g(x)$ that would have a Fourier series with Gibbs phenomenon.

   Provide your answer as (i) a mathematical formula for your function and (ii) a sketch of your function.

   Also, identify the property of your function that would cause Gibbs phenomenon.

Luis Guzman

Math 322 Homework 11 100

1) $u_{tt} - c^2 u_{xx} = 0$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\beta^2$$

$X(x)$ and $T(t)$ both have sin or cos solutions

$x \in (0, L)$

$u_x(0,t) = u_x(L,t) = 0$

$X$ solution is $\cos \beta x$ with $\beta = \frac{n\pi}{L}$ to satisfy Neumann BCs

$$X(x) = \cos \frac{n\pi x}{L}$$

$$T(t) = A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L}$$

$n=0$ case: $\beta = 0$ so $T'' = 0$

has solution $T(t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t$ and $X(x) = \cos 0 = 1$

Combining $X$ and $T$: $u(x,t) = X(x)T(t)$

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L}\right) \cos \frac{n\pi x}{L}}$$

b) $u(x,0) = g(x)$ IC
$u_t(x,0) = h(x)$ IC

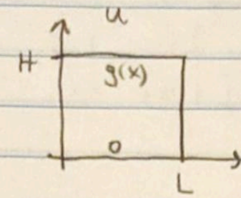
Plug into solution
$$g(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$$
$$\int_0^L \cos\frac{m\pi x}{L} g(x)\,dx = \int_0^L A_n \cos\frac{m\pi x}{L}\left(\sum_{n=1}^{\infty} \cos\frac{n\pi x}{L} + \frac{1}{2}A_0\right)dx$$
$$\int_0^L \cos\frac{m\pi x}{L} g(x)\,dx = A_n\frac{L}{2}$$
$$\boxed{A_n = \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L} g(x)\,dx} \longrightarrow \boxed{A_n = \hat{g}_n}$$
$$u_t(x,t) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty}\left(-\frac{n\pi c}{L}A_n \sin\frac{n\pi ct}{L} + \frac{n\pi c}{L}B_n\cos\frac{n\pi ct}{L}\right)\cos\frac{n\pi x}{L}$$
$$h(x) = u_t(x,0) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\cos\frac{n\pi x}{L}$$

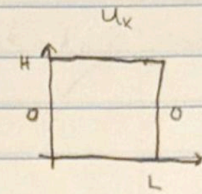
Same case as above $A_n$'s, (divided by $\frac{n\pi c}{L}$)
$$\boxed{\begin{aligned} B_n &= \frac{2}{n\pi c}\int_0^L \cos\frac{n\pi x}{L} g(x)\,dx \\ B_0 &= \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L} g(x)\,dx \end{aligned}} \longrightarrow \boxed{\begin{aligned} \hat{B}_n &= \frac{L}{n\pi c}\hat{h}_n \\ \hat{B}_0 &= \hat{h}_0 \end{aligned}}$$

c) $g(x) = 1 - \frac{1}{2}\cos\frac{2\pi x}{L}$ $\hat{g}_0 = 2$, $\hat{g}_1 = 0$, $\hat{g}_2 = -\frac{1}{2}$ all others $= 0$
$h(x) = 0$ $\hat{h}_n = 0$
$$\boxed{u(x,t) = 1 - \frac{1}{2}\cos\frac{2\pi ct}{L}\cos\frac{2\pi x}{L}}$$

2) $u_{xx} + u_{yy} = 0$ PDE

$$\left.\begin{aligned} u(x,0) &= 0 \\ u(x,H) &= g(x) \\ u_x(0,y) = u_x(L,y) &= 0 \end{aligned}\right\} \text{BCs}$$




a) $u = X(x)\,Y(y) \rightarrow \dfrac{X''}{X} + \dfrac{Y''}{Y} = 0$

$X'' = -\beta^2 X \qquad Y'' = \beta^2 Y \qquad$ do $X$ (hom. BCs) First.

$X(x) = \cos\dfrac{n\pi x}{L}$ with $n = 0, 1, 2, 3 \dots \qquad \beta = \dfrac{n\pi}{L}$

$Y$ solns: $Y(y) = \cosh \beta x$ or $\sinh \beta x$

$u(x,0)$ requires $Y(y) = \sinh \beta y = \sinh \dfrac{n\pi y}{L}$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \hat{u}_n \cos\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}}$$

BC for $y=H$: $\quad g(x) = \sum_{n=1}^{\infty} \hat{u}_n \cos\dfrac{n\pi x}{L} \sinh\dfrac{n\pi H}{L}$

So $\quad \hat{g}_n = \hat{u}_n \sinh\dfrac{n\pi H}{L}$

$$\frac{2}{L}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx = \hat{u}_n \sinh\frac{n\pi H}{L}$$

$$\rightarrow \boxed{\hat{u}_n = \frac{2}{L\sinh\frac{n\pi H}{L}} \int_0^L g(x)\cos\frac{n\pi x}{L}\,dx}$$

b) $g(x) = \cos\dfrac{\pi x}{L}$

$$\hat{u}_n = \frac{2}{L\sinh\frac{n\pi H}{L}} \int_0^L \cos\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{\sinh\frac{\pi H}{L}} \quad \text{for } n=1 \quad \text{all others} = 0$$

3) $f(x) = x(L-x)$

$f'(x) = -x + (L-x) = -2x$

$f''(x) = -2$

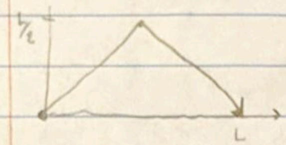
$f$, $f'$, and $f''$ exist and are continuous on $[0, L]$ ✓

$f(0) = 0$

$f(L) = 0$

$f$ satisfies Dirichlet BCs (assumed since we have a Fourier sine series) ✓

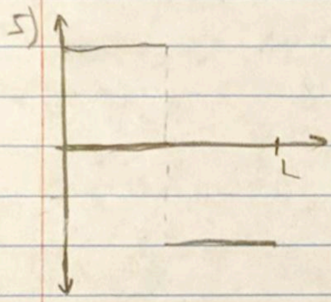
$g(x)$ converges uniformly and therefore also converges point-wise and $L^2$ ✓

4) 

a) $f'$ is not continuous so doesn't converge uniformly

b) $f$ is continuous and $f'$ is piecewise continuous so it converges point wise ✓

c) $\int_a^b |f(x)|^2 dx$ is finite so it converges $L^2$

5) 

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, L/2] \\ -1 & \text{for } x \in [L/2, L] \end{cases}$$

✓

Gibbs phenomenon occurs in piece-wise continuous functions with continuous derivatives. It's fourier series overshoots and "rings" at a discontinuity