Homework 11 Problem 2

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Two coaxial cylinders of inner radius a and outer radius b are separated by an insulating material of susceptibility χ_m. A current I flows down the inner conductor and returns through the outer one; assume that in both cases the current distributes itself uniformly throughout the volume. Assume the outer cylinder is infinitesimally thin. Find the magnetic field between the two conductors. Check your answer by finding all the bound currents and the magnetization and show that they and the free current generate the correct field.

First, solve for $\vec{H}$ inside of a.

$$\oint \vec{H} \cdot dl = I_{f,enc}$$
$$2\pi r|H| = \frac{I\pi r^2}{\pi a^2}$$
$$\vec{H} = \frac{Ir}{2\pi a}\hat{\phi}$$

Between a and b:

$$\oint \vec{H} \cdot dl = I_{f,enc}$$
$$2\pi r|H| = I$$
$$\vec{H} = \frac{I}{2\pi r}\hat{\phi}$$

Outside b:

$$\oint \vec{H} \cdot dl = I_{f,enc} = 0$$
$$\vec{H} = \vec{0}$$

Since the material is assumed to be linear, we can use $\vec{B} = \mu_0(1+\chi_m)\vec{H}$ to find $\vec{B}$. Outside of the material, $\vec{B} = \mu_0\vec{H}$, so

$$\boxed{\vec{B} = \begin{cases} \frac{\mu_0 Ir}{2\pi a}\hat{\phi} & \text{for } r < a \\ \frac{\mu_0 I(1+\chi_m)}{2\pi r}\hat{\phi} & \text{for } r \in [a,b] \\ \vec{0} & \text{for } r > b \end{cases}}$$

Now solve for the bound currents.

$$\vec{K}_b = \vec{M} \times \hat{n} = \chi_m \vec{H} \times \hat{n} = \boxed{\begin{cases} \frac{I\chi_m}{2\pi a}\hat{z} & \text{at } r = a \\ -\frac{I\chi_m}{2\pi b}\hat{z} & \text{at } r = b \end{cases}}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \vec{\nabla}(\chi_m \vec{H}) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\chi_m I r}{2\pi r}\right) = \boxed{\vec{0}}$$

Check our solution for $\vec{B}$ using the free and bound currents and the magnetization.

Inside of a, $I_b = 0$ so $\vec{B}$ should stay the same.

$$\oint \vec{B} \cdot dl = \mu_0 I_{enc}$$
$$2\pi r|B| = \frac{\mu_0 I \pi r^2}{\pi a^2}$$
$$\vec{B} = \frac{\mu_0 I r}{2\pi a}\hat{\phi}$$

Between a and b:

$$\oint \vec{B} \cdot dl = \mu_0 I_{f,enc} + \mu_0 I_{b,enc}$$
$$2\pi r|B| = \mu_0 I + \mu_0 \frac{I\chi_m}{2\pi a}(2\pi a)$$
$$\vec{B} = \frac{\mu_0 I}{2\pi r}(1 + \chi_m)\hat{\phi}$$

Outside b:

$$\vec{B} \cdot dl = \mu_0 I_{f,enc} + \mu_0 I_{b,enc}$$
$$2\pi r|B| = \mu_0 I - \mu_0 I + \mu_0 \frac{I\chi_m}{2\pi a}(2\pi a) - \mu_0 \frac{I\chi_m}{2\pi b}(2\pi b)$$
$$\vec{B} = \vec{0}$$

The solutions match everywhere, so we have verified our answer.