Homework 11 Problem 2

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A solid cylindrical straight wire of radius a has a current I flowing down it.

(a) If that current is uniformly distributed over the outer surface of the wire (none is flowing through the "volume" of the wire; it's all surface charge), what is the surface current density $\vec{K}$?

(b) Suppose that current does flow throughout the volume of the wire, in such a way that the volume current density $\vec{J}$ grows quadratically with distance from the central axis, what then is the formula for $\vec{J}$ everywhere in the wire?

For both of these cases, first solve Laplace's equation $\nabla^2 A_z = \mu_0 J_z$ to find the vector potential as a function of s in cylindrical coordinates. Note that you already know how to solve this equation from electrostatics. Then compute $\vec{B}$ from $\vec{\nabla} \times \vec{A}$.

Solution:

a) Since the current is uniform,

$$\vec{K} = \frac{\vec{I}}{L} = \boxed{\frac{I}{2\pi a}\hat{z}}$$
$$\vec{J} = \frac{I}{2\pi a}\delta(r-a)\hat{z}$$

b) For quadratically distributed current,

$$\vec{J} = \alpha r^2 \hat{z}$$

$$I = \int_S \vec{J} \cdot d\vec{a} = \int_S |\vec{J}| da$$
$$= \int_0^{2\pi} d\phi \int_0^a \alpha r^2 r dr$$
$$= 2\pi\alpha \frac{1}{4} a^4$$
$$I = \frac{\pi\alpha}{2} a^4 \implies \alpha = \frac{2I}{\pi a^4}$$
$$\boxed{\vec{J} = \begin{cases} \frac{2Ir^2}{\pi a^4}\hat{z} \text{ for } r < a. \\ \vec{0} \text{ for } r > a. \end{cases}}$$

Now in order to solve Laplace's equation for these two current distributions, I will use somewhat of a trick. I draw analogy to Laplace's equation in electrostatics, treating the current distribution J_z as a "charge" distribution ρ.

$$\nabla^2 A_z = \mu_0 J_z \implies \nabla^2 A_z = -\frac{\rho}{\epsilon_0}$$

Solving for "potential" V will then give us the correct form of vector potential A, since the form of Laplace's equation is identical. Note that in this case, the units of ρ and V are not actually the same as charge distribution and electric potential (hence the quotation marks), but the analogy makes solving Laplace's equation much easier.

We must first find ρ from J_z.

$$\mu_0 J_z = -\frac{\rho}{\epsilon_0} \implies \rho = -\mu_0 \epsilon_0 J_z$$

$$\rho = \begin{cases} \dfrac{-I\mu_0\epsilon_0}{2\pi a}\delta(r-a) \text{ for part a;} \\ \dfrac{-2I\mu_0\epsilon_0}{\pi a^4} r^2 \text{ for part b.} \end{cases}$$

a) ρ is a cylindrical shell "charge" distribution. Solving for V is a simple case of Gauss's law.

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$|E| 2\pi r l = \frac{\rho 2\pi a l}{\epsilon_0 \delta(r-a)}$$

$$\vec{E} = \begin{cases} \dfrac{-\rho a}{\epsilon_0 r \delta(r-a)} \hat{r} \text{ outside;} \\ \vec{0} \text{ inside.} \end{cases}$$

Outside of the wire:

$$V = -\int \vec{E} \cdot d\vec{l} = \int_{\infty}^{r} \frac{\rho a}{\epsilon_0 r \delta(r-a)} = -\frac{\mu_0 I}{2\pi} \ln(r) + V_0$$

Inside of the wire:

$$V = -\int \vec{E} \cdot d\vec{l} = V(a) - \int_a^r \vec{0} \cdot d\vec{l} = -\frac{\mu_0 I}{2\pi} \ln(a) + V_0$$

Since Laplace's equation is identical for A_z and both A_x and A_y don't contribute, we know that A must have the same form as V.

$$\boxed{A = \begin{cases} \left(-\dfrac{\mu_0 I}{2\pi} \ln(r) + A_0 \right) \hat{z} \text{ outside;} \\ \left(-\dfrac{\mu_0 I}{2\pi} \ln(a) + A_0 \right) \hat{z} \text{ inside.} \end{cases}}$$

So then,

$$\vec{B} = \nabla \times \vec{A} = \boxed{\begin{cases} \dfrac{\mu_0 I}{2\pi r} \hat{\phi} \text{ outside;} \\ \vec{0} \text{ inside.} \end{cases}}$$

We can verify the answer with Ampere's law:

$$\oint \vec{B} \cdot d\vec{a} = |B| 2\pi r = \mu_0 I_{enc}$$

$$\vec{B} = \begin{cases} \dfrac{\mu_0 I}{2\pi r} \hat{\phi} \text{ outside;} \\ \vec{0} \text{ inside.} \end{cases}$$