Homework 02

Due at the date and time indicated in Canvas. Please turn in this assignment into the 323 mailbox, which is located just outside room 2103 Chamberlin Hall.

1. (20 points) Problem 9.2
2. (20 points) Problem 9.8
3. (20 points) Problem 9.35
4. (20 points) Problem 9.36

Problem 9.2 Show that the **standing wave** $f(z, t) = A \sin(kz) \cos(kvt)$ satisfies the wave equation, and express it as the sum of a wave traveling to the left and a wave traveling to the right (Eq. 9.6).

Problem 9.8 Equation 9.36 describes the most general **linearly** polarized wave on a string. Linear (or "plane") polarization (so called because the displacement is parallel to a fixed vector $\hat{\mathbf{n}}$) results from the combination of horizontally and vertically polarized waves of the *same phase* (Eq. 9.39). If the two components are of equal amplitude, but *out of phase* by 90° (say, $\delta_v = 0,\ \delta_h = 90°$), the result is a *circularly* polarized wave. In that case:

(a) At a fixed point z, show that the string moves in a circle about the z axis. Does it go *clockwise* or *counterclockwise*, as you look down the axis toward the origin? How would you construct a wave circling the *other* way? (In optics, the clockwise case is called **right circular polarization**, and the counterclockwise, **left circular polarization**.)[3]

(b) Sketch the string at time $t = 0$.

(c) How would you shake the string in order to produce a circularly polarized wave?

Luis Guzman

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Homework 2

9.2) $f(z,t) = A\sin(Kz)\cos(Kvt)$

$$\frac{\partial}{\partial z} f = AK\cos(Kz)\cos(Kvt)$$

$$\frac{\partial^2}{\partial z^2} f = -AK^2\sin(Kz)\cos(Kvt)$$

$$\frac{\partial}{\partial t} f = AKv\sin(Kz)\sin(Kvt)$$

$$\frac{\partial^2}{\partial t^2} f = -AK^2v^2\sin(Kz)\cos(Kvt)$$

Wave equation: $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial F}{\partial t^2}$ is satisfied

$$-AK^2\sin(Kz)\cos(Kvt) = \frac{1}{v^2}\left(-AK^2v^2\sin(Kz)\cos(Kvt)\right)$$

$$f(z,t) = A\sin(Kz)\cos(Kvt)$$

$$= A\frac{1}{2}\left[\sin(Kz+Kvt) + \sin(Kz-Kvt)\right]$$

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$$\boxed{f(z,t) = \frac{A}{2}\left[\sin(K(z+vt)) + \sin(K(z-vt))\right]}$$

9.8) a) $f(z,t) = Ae^{i(Kz-wt)}\hat{x} + Ae^{i(Kz-wt+\delta_h)}\hat{y}$

set $z=0$ $= A\left(e^{-iwt}\hat{x} + e^{i(-wt+\delta_h)}\right)$

take real part.

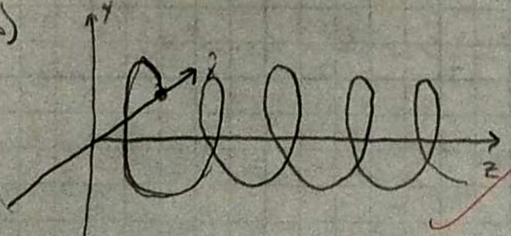
$$= A\left(\cos(-wt)\hat{x} + \cos(-wt+\delta_h)\hat{y}\right)$$

if $\delta_h = \frac{\pi}{2}$ $= A\left(\cos(-wt)\hat{x} - \sin(-wt)\hat{y}\right)$

... which is the eqution of a circle centered at $x=0, y=0$ w/ radius A

it rotates counter clockwise. a clockwise polarization needs $\delta_h = -\frac{\pi}{2}$

b)



wavelength: $\frac{2\pi}{K}$

c) I would rotate it in a circle around the z axis.

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Problem 9.35 Suppose

$$\mathbf{E}(r,\theta,\phi,t) = A\frac{\sin\theta}{r}\left[\cos(kr-\omega t) - (1/kr)\sin(kr-\omega t)\right]\hat{\boldsymbol{\phi}}, \quad \text{with } \frac{\omega}{k} = c.$$

(This is, incidentally, the simplest possible **spherical wave**. For notational convenience, let $(kr - \omega t) \equiv u$ in your calculations.)

(a) Show that **E** obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.

(b) Calculate the Poynting vector. Average **S** over a full cycle to get the intensity vector **I**. (Does it point in the expected direction? Does it fall off like r^{-2}, as it should?)

(c) Integrate $\mathbf{I}\cdot d\mathbf{a}$ over a spherical surface to determine the total power radiated. [*Answer:* $4\pi A^2/3\mu_0 c$]

9.35) a) $\nabla \cdot E = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\cancel{E_\varphi} = 0$ ✓

$$\nabla \times E = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\varphi)\,\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(r E_\varphi)\,\hat{\theta}$$

$$= \frac{\cos\theta}{r\sin\theta}E_\varphi\,\hat{r} - \left(\frac{E_\varphi}{r} + \frac{\partial E}{\partial r}\right)\hat{\theta}$$

$$= \frac{A\cos\theta}{r^2}\left(\cos u - \frac{\sin u}{kr}\right) + \frac{A\cos\theta}{r^2}\left(\cos u - \frac{\sin u}{kr}\right)\hat{r} - \left(\frac{A\sin\theta}{r^2}\left(\cos u - \frac{\sin u}{kr}\right) - \frac{A\sin\theta}{r^2}\left(\cos u - \frac{\sin u}{kr}\right)\right.$$
$$\left. + \frac{A\sin\theta}{r}\left(-k\sin u - \frac{\cos u}{r} + \frac{\sin u}{kr^2}\right)\right)\hat{\theta}$$

$$= \frac{2A\cos\theta}{r^2}\left(\cos u - \frac{\sin u}{kr}\right)\hat{r} - \frac{A\sin\theta}{r}\left(-k\sin u - \frac{\cos u}{r} + \frac{\sin u}{kr^2}\right)\hat{\theta}$$

$$B = \int \nabla \times E \, dt$$

$$= \frac{2A\cos\theta}{r^2}\left(\frac{-1}{\omega}\sin u - \frac{1}{\omega kr}\cos u\right)\hat{r} - \frac{A\sin\theta}{r}\left(\frac{-k}{\omega}\cos u + \frac{1}{\omega r}\sin u + \frac{1}{\omega kr^2}\cos u\right)\hat{\theta}$$

$$\boxed{B = -\frac{2A\cos\theta}{\omega r^2}\left(\sin u + \frac{\cos u}{kr}\right)\hat{r} + \frac{A\sin\theta}{\omega r}\left(-k\cos u + \frac{\sin u}{r} + \frac{\cos u}{kr^2}\right)\hat{\theta}}$$ ✓

$$\nabla \cdot B = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-2A\cos\theta}{\omega}\left(\sin u + \frac{\cos u}{kr}\right)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-A\sin^2\theta}{\omega r}\left(-k\cos u + \frac{\sin u}{r} + \frac{\cos u}{kr^2}\right)\right)$$

$$= \frac{-2A\cos\theta}{\omega r^2}\left(k\cos u - \frac{\cos u}{kr^2} - \frac{\sin u}{r}\right) - \frac{A \cdot 2\sin\theta\cos\theta}{\omega r^2 \sin\theta}\left(-k\cos u + \frac{\sin u}{r} + \frac{\cos u}{kr^2}\right)$$

$$= \frac{-2A\cos\theta}{\omega r^2}\left(k\cos u - k\cos u + \cancel{\frac{\cos u}{kr^2} - \frac{\cos u}{kr^2}} + \frac{\sin u}{r} - \frac{\sin u}{r}\right)$$

$$= 0$$ ✓

$$\nabla \times B = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{-A\sin\theta}{\omega}\left(-k\cos u + \frac{\sin u}{r} + \frac{\cos u}{kr^2}\right)\right) - \frac{\partial}{\partial\theta}\left(\frac{-2A\cos\theta}{\omega r^2}\left(\sin u + \frac{\cos u}{kr}\right)\right)\right]\hat{\varphi}$$

$$= \frac{-A\sin\theta}{r\omega}\left(k^2\sin u + \frac{k\cos u}{r} - \frac{\sin u}{r^2} - \frac{\sin u}{r^2} - \frac{2\cos u}{kr^3}\right) - \frac{2A\sin\theta}{\omega r^3}\left(\sin u + \frac{\cos u}{kr}\right)\hat{\varphi}$$

$$= \frac{-A\sin\theta}{r\omega}\left(k^2\sin u + \frac{k\cos u}{r} - \frac{2\sin u}{r^2} + \frac{2\sin u}{r^2} - \frac{2\cos u}{kr^3} + \frac{2\cos u}{kr^3}\right)\hat{\varphi}$$

$$= -\frac{A\sin\theta}{r\omega}\left(k^2\sin u + \frac{k\cos u}{r}\right)\hat{\varphi}$$

$$u_0\varepsilon_0 \frac{\partial E}{\partial t} = u_0\varepsilon_0 A \frac{\sin\theta}{r}\left(\omega \sin u + \frac{\omega \cos u}{kr}\right)\hat{\varphi} \qquad u_0\varepsilon_0 = \frac{1}{c^2} = \frac{k^2}{\omega^2}$$

$$= \frac{A\sin\theta}{r\omega}\left(k^2 \sin u + \frac{k\cos u}{r}\right)$$

so $\nabla \times B = \mu_0\varepsilon_0 \frac{\partial E}{\partial t}$ ✓

b) $$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}\begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\varphi} \\ 0 & 0 & E_\varphi \\ B_r & B_\theta & 0 \end{vmatrix}$$

$$= -A\frac{\sin\theta}{\mu_0 r}\left(\cos(u) - \frac{\sin u}{kr}\right)\left(\frac{-A\sin\theta}{\omega r}\right)\left(-k\cos u + \frac{\sin u}{r} + \frac{\cos u}{kr^2}\right)\hat{r}$$

$$+ A\frac{\sin\theta}{\mu_0 r}\left(\cos u - \frac{\sin u}{kr}\right)\left(\frac{-2A\cos\theta}{\omega r^2}\right)\left(\sin u + \frac{\cos u}{kr}\right)\hat{\theta}$$

$$= \frac{A^2\sin\theta}{\mu_0\omega r^2}\left[\sin\theta\left(-k\cos^2 u + \frac{\sin u\cos u}{r} + \frac{\cos^2 u}{kr^2} + \frac{\sin u\cos u}{r} - \frac{\sin^2 u}{kr^2} - \frac{\sin u\cos u}{k^2 r^3}\right)\right.$$

$$\left. - \frac{2}{r}\cos\theta\left(\cos u\sin u + \frac{\cos^2 u}{kr} - \frac{\sin^2 u}{kr} - \frac{\sin u\cos u}{k^2 r^2}\right)\hat{\theta}\right]$$

$$\boxed{\vec{S} = \frac{A^2\sin\theta}{\mu_0\omega r^2}\left[\sin\theta\left(-k\cos^2 u + \sin u\cos u\left(\frac{2}{r} - \frac{1}{k^2 r^3}\right) + \frac{1}{kr^2}(1 - 2\sin^2 u)\right)\hat{r} - \frac{2}{r}\cos\theta\left(\cos u\sin u\left(1 - \frac{1}{kr^3}\right) + \frac{1}{kr}(1 - 2\sin^2\theta)\right)\right]}$$

$$\langle S\rangle = \frac{1}{2\pi}\int_0^{2\pi}\vec{S}\,du$$

$$= \frac{A^2\sin\theta}{\mu_0\omega r^2}\left[\left(\sin\theta\left(\frac{-k}{2}\right) + 0 + 0\right)\hat{r} - \frac{2}{r}\cos\theta(0+0)\hat{\theta}\right]$$

$$\boxed{I = \frac{A^2\sin^2\theta\, k}{2\omega r^2\mu_0}\hat{r}}$$

direction ✓

r^{-2} ✓

$$\oint I\cdot da = \int_0^{2\pi} d\varphi\int_0^{\pi}\sin^3\theta\, d\theta \cdot \frac{A^2 k}{2\omega\mu_0}$$

$$= \frac{4\pi A^2 k}{3\omega\mu_0}$$

$$\boxed{= \frac{4\pi A^2}{3c\mu_0}}$$

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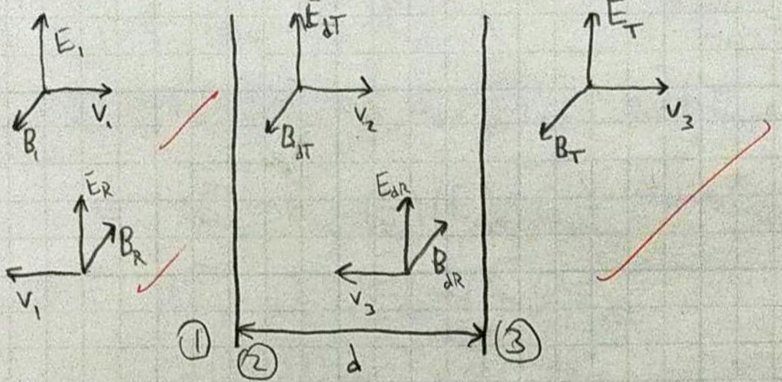
Problem 9.36 Light of (angular) frequency ω passes from medium 1, through a slab (thickness d) of medium 2, and into medium 3 (for instance, from water through glass into air, as in Fig. 9.27). Show that the transmission coefficient for normal incidence is given by

$$T^{-1} = \frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2\left(\frac{n_2\omega d}{c}\right)\right]. \tag{9.199}$$

[*Hint:* To the *left*, there is an incident wave and a reflected wave; to the *right*, there is a transmitted wave; inside the slab, there is a wave going to the right and a wave going to the left. Express each of these in terms of its complex amplitude, and relate the amplitudes by imposing suitable boundary conditions at the two interfaces. All three media are linear and homogeneous; assume $\mu_1 = \mu_2 = \mu_3 = \mu_0$.]

9.36)



Boundary Conditions:

$$\varepsilon_i E_i^{\perp} = \varepsilon_j E_j^{\perp} \qquad E_i^{\parallel} = E_j^{\parallel}$$

$$B_i^{\perp} = B_j^{\perp} \qquad \frac{1}{\mu_i} B_i^{\parallel} = \frac{1}{\mu_j} B_j^{\parallel}$$

$$\underline{\tilde{E}}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x} \qquad \underline{\tilde{B}}_I(z,t) = \frac{1}{v_1} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{y}$$

$$\underline{\tilde{E}}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x} \qquad \underline{\tilde{B}}_R(z,t) = \frac{-1}{v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{y}$$

$$\underline{\tilde{E}}_{dT}(z,t) = \tilde{E}_{0dT} e^{i(k_2 z - \omega t)} \hat{x} \qquad \underline{\tilde{B}}_{dT}(z,t) = \frac{1}{v_2} \tilde{E}_{0dT} e^{i(k_2 z - \omega t)} \hat{y}$$

$$\underline{\tilde{E}}_{dR}(z,t) = \tilde{E}_{0dR} e^{i(-k_2 z - \omega t)} \hat{x} \qquad \underline{\tilde{B}}_{dR}(z,t) = \frac{-1}{v_2} \tilde{E}_{0dR} e^{i(-k_2 z - \omega t)} \hat{y}$$

$$\underline{\tilde{E}}_T(z,t) = \tilde{E}_{0T} e^{i(k_3 z - \omega t)} \hat{x} \qquad \underline{\tilde{B}}_T(z,t) = \frac{1}{v_3} E_{0T} e^{(k_3 z - \omega t)} \hat{y}$$

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normal incidence so $E^{\perp} = B^{\perp} = 0$

$$\begin{cases} \tilde{E}_{0I} + \tilde{E}_{0R} = E_{0dT} \\ \frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0I} - \frac{1}{v_1}\tilde{E}_{0R}\right) = \frac{1}{\mu_2}\frac{1}{v_2}\tilde{E}_{0dT} \end{cases} \qquad \begin{cases} e^{ik_2 d}\tilde{E}_{0dT} + \tilde{E}_{0dR} e^{-ik_2 d} = E_{0T} e^{ik_3 d} \\ \frac{1}{\mu_2}\left(\frac{1}{v_2} e^{ik_2 d} E_{0dT} - \frac{1}{v_2} E_{0dR} e^{-ik_2 d}\right) = \frac{1}{\mu_3}\frac{1}{v_3} E_{0T} e^{ik_3 d} \end{cases}$$

$u_1 = u_2 = u_3 = u_0$

See below:

$$\tilde{E}_{0R} = \frac{v_2 - v_1}{v_2 + v_1}\tilde{E}_{0I} = \frac{n_1 - n_2}{n_2 + n_1}\tilde{E}_{0I} \qquad \tilde{E}_{0dR} = \frac{v_3 - v_2}{v_3 + v_2}\tilde{E}_{0dT} e^{2ik_2 d} = \frac{n_3 - n_2}{n_3 + n_2} E_{0dT} e^{2ik_2 d}$$

$$\tilde{E}_{dT} = \frac{2v_2}{v_2 + v_1}\tilde{E}_{0I} = \frac{2n_1}{n_2 + n_1}\tilde{E}_{0I} \qquad \tilde{E}_{0T} = E_{0dT}\left(\frac{n_2}{n_3}\right) e^{ik_2 d} e^{ik_3 d}\left(1 + \frac{n_3 - n_2}{n_3 + n_2} e^{ik_2 d}\right)$$

$$I = \left\langle \frac{1}{\mu} E \times B \right\rangle = \frac{1}{2}\varepsilon v E^2$$

$$\frac{v_2\left(\tilde{E}_{0dT} e^{2ik_2 d} + \tilde{E}_{0dR}\right)}{e^{2ik_2 d}\tilde{E}_{0dT} - E_{0dR}} = v_3 \rightarrow \frac{v_2}{v_3}\left(\tilde{E}_{0dT} e^{2ik_2 d} + E_{0dR}\right) = e^{2ik_2 d}\tilde{E}_{0dT} - E_{0dR} \rightarrow \left(\frac{v_2}{v_3} + 1\right) E_{0dR} = \left(1 - \frac{v_2}{v_3}\right) E_{dT} e$$

$$\frac{1}{v_2} e^{ik_2 d}\tilde{E}_{0dT} - \frac{1}{v_2}\frac{n_3 - n_2}{n_3 + n_2}\tilde{E}_{0dT} e^{2ik_2 d} = \frac{1}{v_3} e^{ik_3 d}\tilde{E}_T \rightarrow \tilde{E}_{0T} = \tilde{E}_{dT}\left(\frac{v_3}{v_2}\right) e^{ik_2 d}\left(1 + \frac{n_3 - n_2}{n_3 + n_2} e^{ik_2 d}\right) e^{ik_3 d}$$

normal incidence so $E^{\perp} = B^{\perp} = 0$

$$①\begin{cases} \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} + \tilde{E}_{0R} \\ \frac{1}{v_1}\tilde{E}_{0I} - \frac{1}{v_1}\tilde{E}_{0R} = \frac{1}{v_2}E_{0T} - \frac{1}{v_2}E_{0R} \end{cases} \qquad ③\begin{cases} \tilde{E}_{0T}e^{ik_2d} + \tilde{E}_{0R}e^{-ik_2d} = E_{0T}e^{ik_3d} \\ \frac{1}{v_2}\tilde{E}_{0T}e^{ik_2d} - \frac{1}{v_2}E_{0R}e^{-ik_2d} = \frac{1}{v_3}\tilde{E}_{0T}e^{ik_3d} \end{cases}$$

$$② \rightarrow \tilde{E}_{0I} - \tilde{E}_{0R} = \left(\frac{v_1}{v_2}\right)(E_{0T} - E_{0R}) \qquad ④ \rightarrow \tilde{E}_{0T}e^{ik_2d} - \tilde{E}_{0R}e^{-ik_2d} = \frac{v_2}{v_3}\tilde{E}_{0T}e^{ik_3d}$$

① + ② $\quad 2\tilde{E}_{0I} = \left(\frac{v_1}{v_2}+1\right)\tilde{E}_{0T} + \left(1-\frac{v_1}{v_2}\right)\tilde{E}_{0R}$

③ + ④ $\quad 2\tilde{E}_{0T}e^{ik_2d} = \left(\frac{v_2}{v_3}+1\right)\tilde{E}_{0T}e^{ik_3d} \rightarrow E_{0T} = \frac{\left(\frac{v_2}{v_3}+1\right)}{2e^{ik_2d}}E_{0T}e^{ik_3d}$

③ − ④ $\quad 2\tilde{E}_{0R}e^{-ik_2d} = \left(1-\frac{v_2}{v_3}\right)\tilde{E}_{0T}e^{ik_3d} \rightarrow E_{0R} = \frac{\left(1-\frac{v_2}{v_3}\right)}{2e^{-ik_2d}}E_{0T}e^{ik_3d}$

$$2E_{0I} = \left(\frac{v_1}{v_2}+1\right)\left(\frac{v_2}{v_3}+1\right)\frac{E_{0T}}{2}\frac{e^{ik_3d}}{e^{ik_2d}} + \left(1-\frac{v_1}{v_2}\right)\left(1-\frac{v_2}{v_3}\right)\frac{E_{0T}}{2}e^{ik_3d}e^{ik_2d}$$

$$E_{0I} = \frac{E_{0T}}{4}e^{ik_3d}\left[e^{-ik_2d}\left(1+\frac{v_2}{v_3}+\frac{v_1}{v_2}+\frac{v_1}{v_3}\right) + e^{ik_2d}\left(1-\frac{v_1}{v_2}-\frac{v_2}{v_3}+\frac{v_1}{v_3}\right)\right]$$

$$= \frac{E_{0T}}{4}e^{ik_3d}\left[\left(1+\frac{v_1}{v_3}\right)\left(e^{ik_2d}+e^{-ik_2d}\right) + \left(\frac{v_2}{v_3}+\frac{v_1}{v_2}\right)\left(e^{-ik_2d}-e^{ik_2d}\right)\right]$$

$$= \frac{E_{0T}}{4}e^{ik_3d}\left[\left(1+\frac{v_1}{v_3}\right)\cos(k_2d) - i\left(\frac{v_2}{v_3}+\frac{v_1}{v_2}\right)\sin(k_2d)\right]$$

$$T^{-1} \equiv \frac{I_I}{I_T} = \frac{\epsilon_1 v_1 E_{0I}^2}{\epsilon_3 v_3 E_{0T}^2} \quad \text{and} \quad \frac{c^2}{n_i^2} = \frac{1}{\mu_0\epsilon_i} \rightarrow \epsilon_i = \frac{n_i^2}{\mu_0 c^2}$$

$$= \frac{n_1^2 n_3 E_{0T}^2}{16 n_3^2 n_1 E_{0T}^2}\left|e^{ik_3d}\left[\left(1+\frac{v_1}{v_3}\right)\cos(k_2d) - i\left(\frac{v_2}{v_3}+\frac{v_1}{v_2}\right)\sin(k_2d)\right]\right|^2$$

$$= \frac{n_1}{16n_3}\left[\left(1+\frac{v_1}{v_3}\right)^2\cos^2(k_2d) - 2i\left(1+\frac{v_1}{v_3}\right)\left(\frac{v_2}{v_3}+\frac{v_1}{v_2}\right)\cos(k_2d)\sin(k_2d) + \left(\frac{v_2}{v_3}+\frac{v_1}{v_2}\right)^2\sin(k_2d)\right]$$

drop imaginary part

$$= \frac{n_1}{16n_3}\left[\left(1+\frac{v_1}{v_3}\right)^2 + \left(-\frac{v_1^2}{v_3^2} - \cancel{2\frac{v_1}{v_3}} - 1 + \frac{v_2^2}{v_3^2} + \frac{v_1^2}{v_2^2} + \cancel{2\frac{v_1}{v_3}}\right)\sin^2(k_2d)\right]$$

$$= \frac{n_1}{16n_3}\left[\left(1+\frac{v_1}{v_3}\right)^2 - \left(1-\frac{v_1^2}{v_2^2}\right)\left(1-\frac{v_2^2}{v_3^2}\right)\sin^2(k_2d)\right]$$

$$= \frac{n_1}{16n_3}\left[\left(1+\frac{n_3}{n_1}\right)^2 - \left(1-\frac{n_2^2}{n_1^2}\right)\left(1-\frac{n_3^2}{n_2^2}\right)\sin^2(k_2d)\right]$$

$$\boxed{= \frac{1}{16n_1n_3}\left[(n_1+n_3)^2 - \frac{(n_1^2-n_2^2)(n_2^2-n_3^2)}{n_2^2}\sin^2(k_2d)\right]}$$

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