Homework 04

Due at the date and time indicated in Canvas. Please turn in this assignment into the 323 mailbox, which is located just outside room 2103 Chamberlin Hall.

1. (20 points) Problem 10.2
2. (20 points) Problem 10.7 (Hint: you may need Eqn. 1.99)
3. (20 points) Problem 10.12
4. (20 points) Problem 10.15

Example 10.1. Find the charge and current distributions that would give rise to the potentials

$$V = 0, \quad \mathbf{A} = \begin{cases} \dfrac{\mu_0 k}{4c}(ct - |x|)^2\,\hat{\mathbf{z}}, & \text{for } |x| < ct, \\ \mathbf{0}, & \text{for } |x| > ct, \end{cases}$$

where k is a constant, and (of course) $c = 1/\sqrt{\epsilon_0 \mu_0}$.

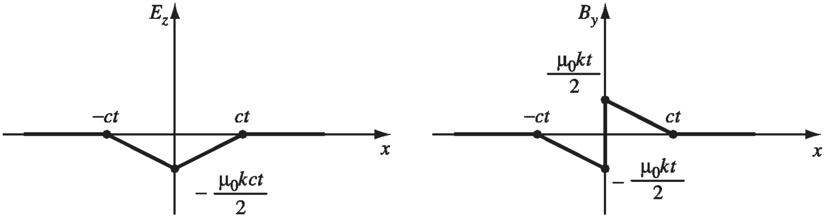

FIGURE 10.1

Problem 10.2 For the configuration in Ex. 10.1, consider a rectangular box of length l, width w, and height h, situated a distance d above the yz plane (Fig. 10.2).

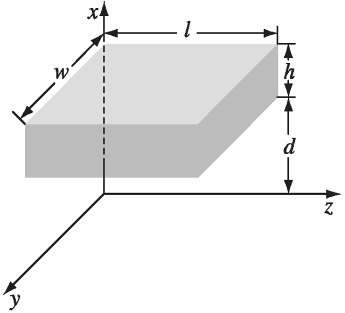

FIGURE 10.2

(a) Find the energy in the box at time $t_1 = d/c$, and at $t_2 = (d + h)/c$.

(b) Find the Poynting vector, and determine the energy per unit time flowing into the box during the interval $t_1 < t < t_2$.

(c) Integrate the result in (b) from t_1 to t_2, and confirm that the increase in energy (part (a)) equals the net influx.

Homework 4	Luis Guzman

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1) $u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu_0} B^2\right)$

$E = -\frac{\mu_0 K}{2}(ct - |x|)\hat{z}$

$B = \frac{\mu_0 K}{2c}(ct - |x|)\hat{y}$

$$E_{tot} = \frac{1}{2}\iiint \varepsilon\left(\frac{\mu_0 K}{2}(ct-|x|)\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 K}{2c}(ct-|x|)\right)^2 dV$$

$$= \frac{\mu_0^2 K^2 w \cdot l}{8}\int_d^{d+h}\left(\varepsilon + \frac{1}{\mu_0 c^2}\right)(ct-|x|)^2\, dx$$

$$= \frac{\varepsilon \mu_0^2 K^2 w \cdot l}{4}\left[\frac{-1}{3}(ct-x)^3\right]_d^{d+h}$$

$$= \frac{-\varepsilon \mu_0^2 K^2 w \cdot l}{12}\left((ct-h-d)^3 - (ct-d)^3\right)$$

$$E_{tot}\Big|_{t_1} = \frac{-\varepsilon \mu_0^2 K^2 w \cdot l}{12}(-h^3 + h^3) \boxed{= 0}$$

$$E_{tot}\Big|_{t_2} = \frac{-\varepsilon \mu_0^2 K^2 w \cdot l}{12}(0 - h^3) = \boxed{\frac{\varepsilon \mu_0^2 K^2 w l h^3}{12}}$$

b) $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$

$\boxed{\vec{S} = -\frac{\mu_0 K^2}{4c}(ct-|x|)^2 \hat{x}}$ For $|x| < ct$

$$\frac{dW}{dt} = \int S \cdot da \boxed{= \frac{\mu_0 K^2 w \cdot l}{4c}(ct-d)^2}$$

c) $$\int_{\frac{d}{c}}^{\frac{d+h}{c}} \frac{\mu K^2 w \cdot l}{4c}(ct-d)^2\, dt$$

$$= \frac{\mu_0 K^2 w \cdot l}{4c}\left[\frac{1}{3c}(ct-d)^3\right]_{d/c}^{\frac{d+h}{c}}$$

$$= \frac{\mu_0 K^2 w \cdot l h^3}{12 c^2}$$

$$\boxed{= \frac{\varepsilon \mu_0^2 K^2 w \cdot l h^3}{12}}$$

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Problem 10.7 A time-dependent point charge $q(t)$ at the origin, $\rho(\mathbf{r}, t) = q(t)\delta^3(\mathbf{r})$, is fed by a current $\mathbf{J}(\mathbf{r}, t) = -(1/4\pi)(\dot{q}/r^2)\,\hat{\mathbf{r}}$, where $\dot{q} \equiv dq/dt$.

(a) Check that charge is conserved, by confirming that the continuity equation is obeyed.

(b) Find the scalar and vector potentials in the Coulomb gauge. If you get stuck, try working on (c) first.

(c) Find the fields, and check that they satisfy all of Maxwell's equations.[3]

10.7) $\rho(\vec{r},t) = q(t)\delta^3(\vec{r})$ $\quad\quad$ $\vec{J}(\vec{r},t) = \frac{-1}{4\pi}\left(\frac{\dot{q}}{r^2}\right)\hat{r}$

a) $\frac{\partial \rho}{\partial t} = \dot{q}(t)\delta^3(\vec{r})$

$-\nabla\cdot\vec{J} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-1}{4\pi}\dot{q}\right) = 0$

$\oiint -\nabla\cdot J = \oint -J\cdot da = \frac{1}{4\pi}\frac{\dot{q}}{R^2}4\pi R^2 = \dot{q}$ ✓ so $-\nabla\cdot J = \dot{q}(t)\delta^3(\vec{r})$ ✓

b) $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{q(t)\delta^3(\vec{r}')}{\mathscr{r}}d\tau'$

$\boxed{V = \frac{q(t)}{4\pi\varepsilon_0 r}}$

$\vec{J}$ points in $\hat{r}$ so $\boxed{\vec{A} = \vec{0}}$ ✓

c) $\boxed{E = \frac{q(t)}{4\pi\varepsilon_0 r^2}\hat{r}}$ $\quad$ $\boxed{B = \vec{0}}$

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$\nabla\cdot E = \frac{q(t)}{4\pi\varepsilon_0}\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{q(t)\delta^3(\vec{r})}{\varepsilon_0} = \frac{\rho}{\varepsilon_0}$ ✓

$\nabla\times E = 0 = -\frac{\partial B}{\partial t}$ ✓ ✓

$\nabla\cdot B = 0$ ✓

$\nabla\times B = 0 = \mu_0\frac{-1}{4\pi}\left(\frac{\dot{q}}{r^2}\hat{r}\right) + \mu_0\varepsilon_0\frac{\dot{q}(t)}{4\pi\varepsilon_0 r^2}\hat{r} = 0$ ✓

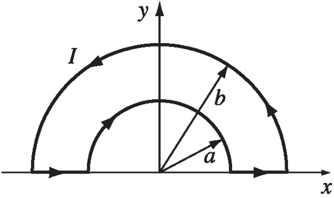


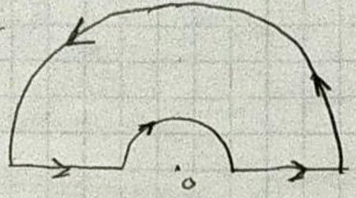
FIGURE 10.5

Problem 10.12 A piece of wire bent into a loop, as shown in Fig. 10.5, carries a current that increases linearly with time:

$$I(t) = kt \quad (-\infty < t < \infty).$$

Calculate the retarded vector potential **A** at the center. Find the electric field at the center. Why does this (neutral) wire produce an *electric* field? (Why can't you determine the *magnetic* field from this expression for **A**?)

10.12



$I(t) = kt \qquad V(x=y=z=0) = 0$

$$A(\vec{r}=\vec{0},t) = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}\,',t_r)}{|\Delta \vec{r}|}\,d\tau'$$

$$J = k t_r \delta(z)\left[\delta(s-a)\hat{\varphi} - \delta(s-b)\hat{\varphi} + \delta(y)\hat{x}\right] \quad \text{for } |x| \in [a,b]$$

$$A(\vec{r}=\vec{0},t) = \frac{k\mu_0}{4\pi}\left[-\int_0^{\pi}\frac{a\, t_r \sin\varphi\,\hat{x}\,d\varphi}{a} + \int_0^{\pi}\frac{b\, t_r \sin\varphi\,\hat{x}\,d\varphi}{b} + 2\int_a^b \frac{t_r\,\hat{x}\,dx}{x}\right]$$

$t_r = t - \frac{x}{c}$

$$= \frac{k\mu_0}{4\pi}\left[+\left(t-\frac{a}{c}\right)\left[\cos\varphi\right]_0^{\pi} - \left(t-\frac{b}{c}\right)\left[\cos\varphi\right]_0^{\pi} + 2\left[t\ln x - \frac{x}{c}\right]_a^b\right]\hat{x}$$

$$= \frac{k\mu_0}{4\pi}\left[-2\left(t-\frac{a}{c}\right) + 2\left(t-\frac{b}{c}\right) + 2t\ln\frac{b}{a}\right]\hat{x}$$

$$\boxed{\vec{A} = \frac{k\mu}{2\pi}\left[\left(\frac{a}{c}-\frac{b}{c}\right) + t\ln\frac{b}{a}\right]\hat{x}}$$

without retarded time $\rightarrow \frac{k\mu_0 t}{2\pi}\ln\frac{b}{a}$ ✓

$$\boxed{E = -\frac{\partial A}{\partial t} = \frac{-k\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)}$$ ✓

The current is varying so the changing B field changes the E field. We need to know $\vec{A}$ everywhere in order to find B. ✓

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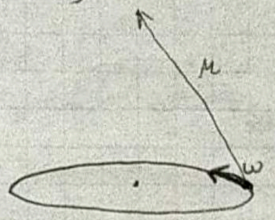
Problem 10.15 A particle of charge q moves in a circle of radius a at constant angular velocity ω. (Assume that the circle lies in the xy plane, centered at the origin, and at time $t = 0$ the charge is at $(a, 0)$, on the positive x axis.) Find the Liénard-Wiechert potentials for points on the z axis.

10.15)



$$V = \frac{1}{4\pi\varepsilon_0} \frac{qc}{c\sqrt{z^2+a^2}}$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{z^2+a^2}}}$$

$$\boxed{A = \frac{\omega a q}{4\pi\varepsilon_0 c^2 \sqrt{z^2+a^2}} \left(\sin \omega t_r \, \hat{y} + \cos \omega t_r \, \hat{x}\right)}$$

$\vec{r}$ is $\perp$ to $\vec{V}$ so $r \cdot v = 0$

$$\vec{V} = \omega a \left(\sin \omega t_r \, \hat{y} + \cos \omega t_r \, \hat{x}\right)$$

$$t_r = t - \frac{\sqrt{z^2+a^2}}{c}$$

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