Physics 531 Homework 9

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1. *Gate fidelity.* In the quantum computing world, the fidelity of a gate U is evaluated as the mean square overlap of the experimentally obtained state with the target state, averaged over initial states that point along the cardinal directions of the Bloch sphere:

$$F_{\text{avg}} = \frac{1}{6} \sum_{|\alpha\rangle \in \nu} |\langle \psi_{\text{ideal}} | U | \alpha \rangle|^2$$

where the set of states ν is as follows:

$$\begin{aligned} |x_\pm\rangle &= \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \\ |y_\pm\rangle &= \frac{|0\rangle \pm i\,|1\rangle}{\sqrt{2}} \\ |z_+\rangle &= |0\rangle \\ |z_-\rangle &= |1\rangle \end{aligned}$$

(a) One source of infidelity is miscalibration of the control pulse length, giving rise to a systematic overrotation or underrotation in state space. Consider a miscalibrated $\pi/2$ y-pulse, i.e., a gate that realizes the rotation $R_y(\pi/2 + \delta)$ instead of the desired rotation $R_y(\pi/2$, where δ is a small overrotation angle. What is the infidelity of the gate?

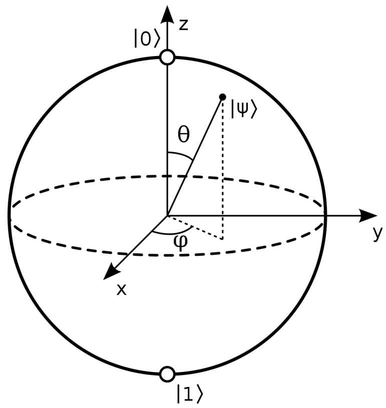
(b) Now consider a gate where the frequency of the control pulse is detuned from resonance by an amount Δ. Assume the gate duration is τ and that the rate of rotation of the state vector is uniform throughout the duration of the gate (i.e., fixed Rabi frequency). What is the infidelity of a nominal $R_y(\pi/2$ rotation in terms of Δ and τ?

Solution:

(a) We can describe the rotation with the unitary rotation operator:

$$\begin{aligned} U &= R_y(\pi/2 + \delta) \\ &= \begin{pmatrix} \cos\frac{\pi}{4} + \frac{\delta}{2} & -\sin\frac{\pi}{4} + \frac{\delta}{2} \\ \sin\frac{\pi}{4} + \frac{\delta}{2} & \cos\frac{\pi}{4} + \frac{\delta}{2} \end{pmatrix} \\ &= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \end{aligned}$$

where $c \equiv \cos\frac{\pi}{4} + \frac{\delta}{2}$ and $s \equiv \sin\frac{\pi}{4} + \frac{\delta}{2}$.



By imagining the Bloch sphere, we can determine what ψ_{ideal} is for each of the states aligned with the coordinate axes. A rotation of $\pi/2$ about the y axis maps x_+ to z_-, z_- to x_-, and so on. The states aligned with the y axis are unchanged. We can then calculate the gate infidelity for each state using $|\langle\psi_{\text{ideal}}|U|\alpha\rangle|^2$.

$$
\begin{aligned}
\langle z_-|U|x_+\rangle &= \langle 1| \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} c-s \\ c+s \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}(c+s) \\
|\langle z_-|U|x_+\rangle|^2 &= \frac{1}{2}(c+s)^2 \\
&= \frac{1}{2}(1+\cos\delta)
\end{aligned}
$$

The states x_-, z_+, and z_- will have the same infidelity (when squared) due to symmetry. Since y_+ and y_- are unchanged, their infidelity is equal to 1. The average gate fidelity is then equal to

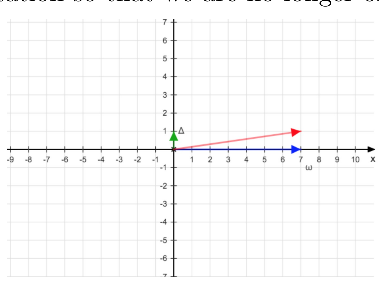
$$
\begin{aligned}
F_{\text{avg}} &= \frac{2}{6}(1+\cos\delta) + \frac{2}{6} \\
&= \frac{1}{3}(2+\cos\delta)
\end{aligned}
$$

and the infidelity is just

$$
\boxed{1 - F_{\text{avg}} = \frac{1}{3}(1-\cos\delta)}
$$

We can check that when $\delta = 0$, $1 - F_{\text{avg}} = 0$ $\surd$

(b) The detuning alters the axis of rotation so that we are no longer on the y axis.



Our rotation operator becomes

$$
\begin{aligned}
u\cdot\sigma &= \frac{\omega_1}{\omega_R}\sigma_y + \frac{\Delta}{\omega_2}\sigma_x \\
R_{\vec{u}}(\frac{\pi}{2}) &= \cos\frac{\pi}{4}\mathbb{1} - i\sin\frac{\pi}{4}\vec{\sigma}\cdot\vec{u} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i(\frac{\Delta}{\omega_R} - i\frac{\omega_1}{\omega_2}) \\ -i(\frac{\Delta}{\omega_R} + i\frac{\omega_1}{\omega_2}) & 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \frac{-\omega_1 - i\Delta}{\omega_R} \\ \frac{\omega_1 - i\Delta}{\omega_R} & 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -a^* \\ a & 1 \end{pmatrix}
\end{aligned}
$$

where $a \equiv \frac{\omega_1 - i\Delta}{\omega_R}$ and a^* is the complex conjugate of a.
Applying this operator to each of our coordinate axis states, we can calculate the infidelity.

$$
\begin{aligned}
\langle z_-|R_{\vec{u}}(\frac{\pi}{2})|x_+\rangle &= \langle 1|\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -a^* \\ a & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1-a^* \\ 1+a \end{pmatrix} = \boxed{\frac{1}{2}(a+1)} \\
\langle z_+|R_{\vec{u}}(\frac{\pi}{2})|x_-\rangle &= \langle 0|\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -a^* \\ a & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1+a^* \\ a-1 \end{pmatrix} = \boxed{\frac{1}{2}(a^*+1)} \\
\langle x_+|R_{\vec{u}}(\frac{\pi}{2})|z_+\rangle &= \begin{pmatrix} 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -a^* \\ a & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ a \end{pmatrix} = \boxed{\frac{1}{2}(1+a)} \\
\langle x_-|R_{\vec{u}}(\frac{\pi}{2})|z_-\rangle &= \begin{pmatrix} 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -a^* \\ a & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} -a^* \\ 1 \end{pmatrix} = \boxed{\frac{1}{2}(-1-a^*)} \\
\langle y_+|R_{\vec{u}}(\frac{\pi}{2})|y_+\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -a^* \\ a & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} -1-ia^* \\ a+i \end{pmatrix} = \boxed{\frac{1}{2\sqrt{2}}(-ia^*+ia)} \\
\langle y_-|R_{\vec{u}}(\frac{\pi}{2})|y_-\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -a^* \\ a & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} -1+ia^* \\ a-i \end{pmatrix} = \boxed{\frac{1}{2\sqrt{2}}(ia^*-ia)}
\end{aligned}
$$

The average gate fidelity is then

$$
\begin{aligned}
F_{\text{avg}} &= \frac{1}{6}\sum_{|\alpha\rangle\in\nu} |\langle\psi_{\text{ideal}}|U|\alpha\rangle|^2 \\
&= \frac{1}{24}[(a+1)^2 + (a^*+1)^2 + (1+a)^2 + (-a^*-1)^2 + \frac{1}{2}(-ia^*+ia)^2 + \frac{1}{2}(ia^*-ia)^2] \\
&= \frac{1}{24}[2(a+1)^2 + 2(a^*+1)^2 + (-ia^*+ia)^2] \\
&= \frac{1}{24}[(2a^2+4a+2) + (2(a^*)^2 + 4a^* + 2) + (-a^* + aa^* - a)] \\
&= \frac{1}{24}[2\frac{\omega_1^2 - 2i\omega_1\Delta - \Delta^2}{\omega_R^2} + 2\frac{\omega_1^2 + 2i\omega_1\Delta - \Delta^2}{\omega_R} + 3\frac{\omega_1}{\omega_R} + \frac{\omega_1^2+\Delta^2}{\omega_R} + 4] \\
&= \frac{1}{24}[4\frac{\omega_1^2-\Delta^2}{\omega_R^2} + 3\frac{\omega_1}{\omega_R} + \frac{\omega_1^2+\Delta^2}{\omega_R} + 4]
\end{aligned}
$$

$$
\boxed{F_{\text{avg}} = \frac{1}{24\omega_R}[4\frac{\omega_1^2-\Delta^2}{\omega_R} + 3\omega_1 + \omega_1^2 + \Delta^2 + 4\omega_R]}
$$

With $\omega_R = \sqrt{\Delta^2+\omega_1^2}$ and $\omega_1 = \frac{\pi}{2\tau}$.

2. *Rotating wave approximation.* Consider the Hamiltonian

$$H = -\frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\omega_1}{2}\cos\omega t\ \sigma_x,$$

where ω_1 again parameterizes the strength of the small oscillating field (in general we will have $\omega_1 \ll \omega_0$). This is nearly identical to the Rabi Hamiltonian you encountered in several problems last week, with the difference that the excitation field is now linearly polarized (there is no $\sin\omega t\ \sigma_y$ component).

(a) Use the time-dependent unitary transformation $U = e^{(-i\omega t/2)\sigma_z}$ to transform to a reference frame rotating at angular frequency ω, the frequency of the excitation field. Derive the effective Hamiltonian H_{eff} in this rotating frame. *Hint.* In contrast to the effective Hamiltonian in last week's problems, this one will no longer be time-independent.

(b) Consider a spin initially in the ground state $|0\rangle$. Integrate the Hamiltonian to determine the time-evolved state (in the rotating frame). You are free to do this analytically, but a numerical approach will be simpler. In detail: discretize the time axis and for each small time step treat the Hamiltonian as fixed so that you can evolve the state in the usual way using the instantaneous unitary evolution operator. For concreteness, take $\omega = \omega_0$ (resonant drive) and $\omega_1 = 0.1\omega_0$.

Solution:

Our Hamiltonian is

$$\begin{aligned} H &= -\frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\omega_1}{2}\cos\omega t\ \sigma_x \\ &= \begin{pmatrix} -\frac{\hbar\omega_0}{2} & \frac{\hbar\omega_1}{2}\cos\omega t \\ \frac{\hbar\omega_1}{2}\cos\omega t & \frac{\hbar\omega_0}{2} \end{pmatrix} \end{aligned}$$

and our unitary rotation operator is

$$\begin{aligned} U &= e^{(-i\omega t/2)\sigma_z} \\ &= \begin{pmatrix} e^{(-i\omega t/2)} & 0 \\ 0 & e^{(i\omega t/2)} \end{pmatrix} \end{aligned}$$

The Schrodinger equation in a rotating frame is the same as before (see Homework 8).

$$i\hbar\frac{d}{dt}\tilde{\psi} = \frac{\hbar\omega}{2}\sigma_z\tilde{\psi} + \tilde{H}\tilde{\psi}$$

Our effective Hamiltonian is then

$$H_{\text{eff}} = \tilde{H} + \frac{\hbar\omega}{2}\sigma_z$$

which can can write more concretely by solving for $\tilde{H}$

$$\begin{aligned} \tilde{H} &= U\hat{H}U^\dagger \\ &= \begin{pmatrix} e^{(-i\omega t/2)} & 0 \\ 0 & e^{(i\omega t/2)} \end{pmatrix} \begin{pmatrix} -\frac{\hbar\omega_0}{2} & \frac{\hbar\omega_1}{2}\cos\omega t \\ \frac{\hbar\omega_1}{2}\cos\omega t & \frac{\hbar\omega_0}{2} \end{pmatrix} \begin{pmatrix} e^{(i\omega t/2)} & 0 \\ 0 & e^{(-i\omega t/2)} \end{pmatrix} \\ &= \begin{pmatrix} e^{(-i\omega t/2)} & 0 \\ 0 & e^{(i\omega t/2)} \end{pmatrix} \begin{pmatrix} -\frac{\hbar\omega_0}{2}e^{(i\omega t/2)} & \frac{\hbar\omega_1}{2}\cos\omega t e^{(-i\omega t/2)} \\ \frac{\hbar\omega_1}{2}\cos\omega t e^{(i\omega t/2)} & \frac{\hbar\omega_0}{2}e^{(-i\omega t/2)} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\hbar\omega_0}{2} & \frac{\hbar\omega_1}{2}\cos\omega t e^{(-i\omega t)} \\ \frac{\hbar\omega_1}{2}\cos\omega t e^{(i\omega t)} & \frac{\hbar\omega_0}{2} \end{pmatrix} \end{aligned}$$

$$\boxed{H_{\text{eff}} = \frac{\hbar}{2}\begin{pmatrix} (\omega - \omega_0) & -\omega_1\cos(\omega t)e^{(-i\omega t)} \\ \omega_1\cos(\omega t)e^{(i\omega t)} & -(\omega - \omega_0) \end{pmatrix}}$$

Starting with the differential equation

$$i\hbar\frac{d}{dt}\hat{U} = H\hat{U}$$

we can solve for the evolution of our initial state by writing $\hat{U}$ in terms of its Dyson series.

$$U_n = \left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n H(t_1)\cdots H(t_n)$$

$$U(t,t') = 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{i}{\hbar}\right)^n}{n!} \int_{t'}^{t} dt_1 \cdots \int_{t'}^{t_{n-1}} dt_n \mathcal{T}[H(t_1)\cdots H(t_n)]$$

where, for instance

$$\mathcal{T}[H(t_1)\cdots H(t_n)] = \begin{cases} H(t_1)H(t_2) & \text{if } t_2 > t_1 \\ H(t_2)H(t_1) & \text{if } t_1 > t_2 \end{cases}$$

for n=2. Therefore our initial state evolves as

$$|\tilde{0}\rangle = U(t,0)\,|0\rangle$$

$$\boxed{|\tilde{0}\rangle = |0\rangle\, \mathcal{T} e^{\frac{-i}{\hbar}\int_0^t H(t_1)dt_1}}$$

4. Consider a quantum harmonic oscillator with characteristic frequency 0. The system is in thermal equilibrium at temperature T. The oscillator is described by the following density matrix:

$$\hat{\rho} = A\exp\left(\frac{-\hat{H}}{k_B T}\right),$$

where $\hat{H}$ is the usual harmonic oscillator Hamiltonian and k_B is Boltzmann's constant. Working in the Fock (photon number) basis:

(a) Find the diagonal elements of $\hat{\rho}$.

(b) Determine the normalization constant A.

(c) Calculate the expectation value of energy $\langle E\rangle$.

Solution:

(a) In the Fock basis, the Hamiltonian of the quantum harmonic oscillator is

$$\hat{H} = \hbar\omega \begin{pmatrix} 1/2 & & & 0 & \\ & 3/2 & & & \\ & & 5/2 & & \\ & & & 7/2 & \\ 0 & & & & \ddots \end{pmatrix}$$

Since $\hat{H}$ is diagonal, solving for $\hat{\rho}$ is simply a matter of multiplying by the constant terms and exponentiating the diagonal entries.

$$\boxed{\hat{\rho} = A \begin{pmatrix} e^{\frac{-\hbar\omega}{2k_B T}} & & & 0 & \\ & e^{\frac{-3\hbar\omega}{2k_B T}} & & & \\ & & e^{\frac{-5\hbar\omega}{2k_B T}} & & \\ & & & e^{\frac{-7\hbar\omega}{2k_B T}} & \\ 0 & & & & \ddots \end{pmatrix}}$$

(b) We can determine the normalization constant by setting the trace of ρ equal to 1.

$$\begin{aligned}\mathrm{Tr}[\rho] &= 1A(e^{\frac{-\hbar\omega}{2k_BT}} + e^{\frac{-3\hbar\omega}{2k_BT}} + e^{\frac{-5\hbar\omega}{2k_BT}} + e^{\frac{-7\hbar\omega}{2k_BT}} + \cdots) \\ &= 1\end{aligned}$$

So then

$$\begin{aligned}A &= \frac{1}{\sum_{n=0}^{\infty} e^{\frac{-\hbar\omega}{2k_BT}}(e^{\frac{-\hbar\omega}{k_BT}})^n} \\ &= \frac{1-e^{\frac{-\hbar\omega}{k_BT}}}{e^{\frac{-\hbar\omega}{2k_BT}}} \quad \text{for } |e^{\frac{-\hbar\omega}{k_BT}}| < 1 \\ &= e^{\frac{\hbar\omega}{2k_BT}} - e^{\frac{-\hbar\omega}{2k_BT}} \\ &\boxed{A = 2\sinh\frac{\hbar\omega}{2k_BT}}\end{aligned}$$

(c) The expected energy is then

$$\begin{aligned}\langle E\rangle &= \sum_{i,j} H_{ij}\rho_{ji} \\ &= 2\hbar\omega\sinh\frac{\hbar\omega}{2k_BT}\sum_{n=0}^{\infty}(\frac{1}{2}+n)e^{\frac{-\hbar\omega}{2k_BT}}(e^{\frac{-\hbar\omega}{k_BT}})^n \\ &= 2\hbar\omega\sinh\frac{\hbar\omega}{2k_BT}e^{\frac{-\hbar\omega}{k_BT}}\left(\frac{1}{2(1-e^{\frac{-\hbar\omega}{k_BT}})} + \frac{e^{\frac{-\hbar\omega}{k_BT}}}{(1-e^{\frac{-\hbar\omega}{k_BT}})^2}\right) \quad \text{for } |e^{\frac{-\hbar\omega}{k_BT}}| < 1 \\ &= 2\hbar\omega\left(e^{\frac{-\hbar\omega}{2k_BT}} - e^{\frac{-3\hbar\omega}{2k_BT}}\right)\left(\frac{1+e^{\frac{-\hbar\omega}{k_BT}}}{2(1-e^{\frac{-\hbar\omega}{k_BT}})^2}\right) \\ &\boxed{\langle E\rangle = \hbar\omega e^{\frac{-\hbar\omega}{2k_BT}}\left(\frac{1+e^{\frac{-\hbar\omega}{k_BT}}}{1-e^{\frac{-\hbar\omega}{k_BT}}}\right)}\end{aligned}$$